Restricted symmetry in graphs and maps

Primož Potočnik

Department of Mathematics
Faculty of Mathematics and Physics
University of Ljubljana

also affiliated with

Institute of Mathematics, Physics and Mechanics
Ljubljana

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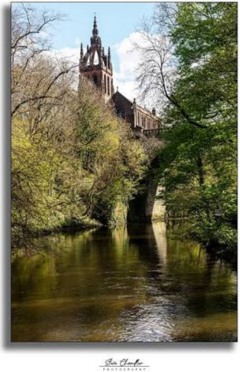
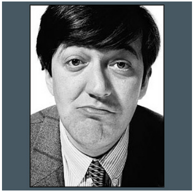


What is restricted symmetry

We like symmetry.

However, some asymmetry may be even more pleasing.



We will be interested in objects that are highly symmetrical, but with a specific kind of symmetry prohibited.

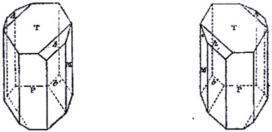
Such a phenomenon will be called restricted symmetry.

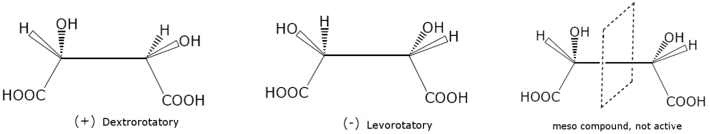
CHIRALITY

Louis Pasteur

In 1848, a graduate student (named Louis Pasteur) was growing crystals of a salt of tartaric acid.

He noticed that they come in two forms:





Stereoisomers of tartaric acid: two chiral enantiomers, and a reflexible diastereomer.

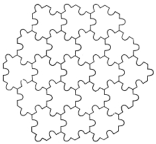
Lord Kelvin

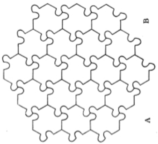
The term chiral was coined 45 years later, by William Thomson, better known as 1st Baron Kelvin of Largs.

On Tuesday, May 16, 1893, he delivered a lecture *The molecular tactics of a crystal* to the Oxford University Junior Scientific Club.

There, he defined:

"I call any geometrical figure . . . chiral . . . if its image in a plane mirror . . . cannot be brought to coincide with itself."





Chirality in maps

Map := 'nice' embedding of a graph into a closed surface.

Chirality only makes sense on orientable surfaces.

A map is chiral if it has no orientation reversing automorphisms.

Such maps are easy to find (embed a graph with trivial automorphism group). In fact:

- Richmond–Wormald (1995): Almost all maps on any fixed surface have trivial automorphism group — hence chiral.
- Drmota–Nedela (2012): Almost all maps are chiral.

Rotary maps

Situation becomes more interesting if we require maximal orientation preserving symmetry.

More precisely, we require the orientation preserving group $\mathrm{Aut}^+(M)$ to act transitively on the arcs of the map.

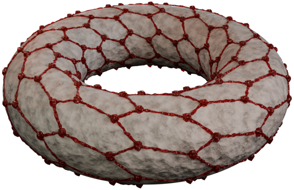
Such maps are called orientably regular by Conder (and several others) and rotary by Wilson (and his followers).

If a rotary map admits a reflection, it is regular. If not: chiral.

Plato ($\cong 400$ BC): All rotary maps on the sphere are regular.

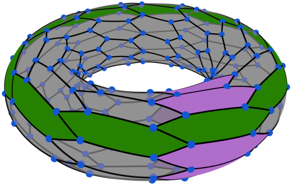
What about higher genera?

Toroidal $\{6, 3\}$-maps

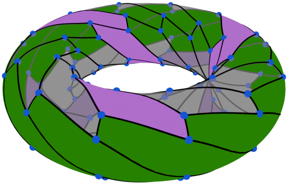


- All are vertex-transitive;
- All admit a $180°$-rotation about a centre of a hexagon;
- Typically no additional symmetry exists, but rotary ones (reflexible and chiral) exist and can be classified.

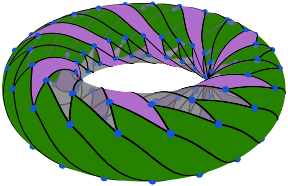
Rotary toroidal $\{6,3\}$-maps



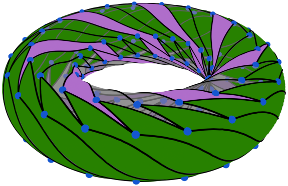
$\begin{pmatrix} 10 & 0 \\ 0 & 10 \end{pmatrix}$ regular



$\begin{pmatrix} 12 & 4 \\ 0 & 4 \end{pmatrix}$ regular

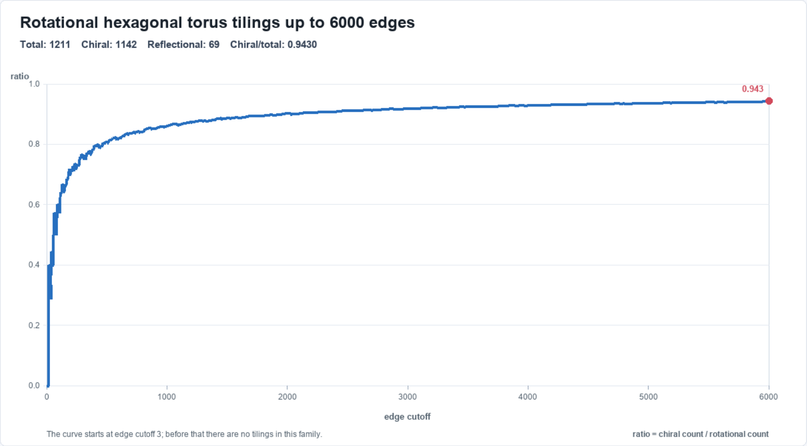


$\begin{pmatrix} 21 & 6 \\ 0 & 3 \end{pmatrix}$ rotary chiral



$\begin{pmatrix} 21 & 12 \\ 0 & 3 \end{pmatrix}$ rotary chiral

Proportion of chiral among rotary $\{6, 3\}$ toroidal maps



Chiral maps of higher genus – hyperbolic type

For genera ≥ 2, finding chiral maps is somewhat trickier. None exist for genera $2, \ldots, 6$, 9, $13, \ldots$.

Conder–Širáň–Tucker (2010): There are infinitely many orientable surfaces that support no chiral rotary maps.

But if we are not interested in the genus, they exist for every hyperbolic type (k, m).

- Conjectured by Singerman in 1992;
- Announced by Hučíková, Nedela, and Širáň (preprint 2012);
- Proved for hypermaps by Jones in 2013 (alternative proof by Širáň and Read in 2024).

Chiral maps of hyperbolic type

Extended triangle group: $\Delta^*(k,m,2) = \langle a,b,c \mid a^2, b^2, c^2, (ab)^k, (bc)^m, (ac)^2 \rangle$

hyperbolic: $\frac{1}{k} + \frac{1}{m} < \frac{1}{2}$ $\qquad \leq \mathrm{Isom}(\mathbb{H})$, preserves $\{k,m\}$-tiling.

$$\begin{tikzcd}
\Delta^* \arrow[d, dash, "2"] \\
\Delta = \langle ab, bc \rangle \arrow[d, dash, "<\infty"] \\
T \trianglelefteq \Delta \arrow[d, dash, "<\infty"] \\
L
\end{tikzcd}$$

$T \trianglelefteq \Delta$ torsion free

possibly $T \trianglelefteq \Delta^*$ if $\mathbb{H}/T$ reflexible

$L \leftarrow$ want to find s.t. $N_{\Delta^*}(L) = \Delta$.

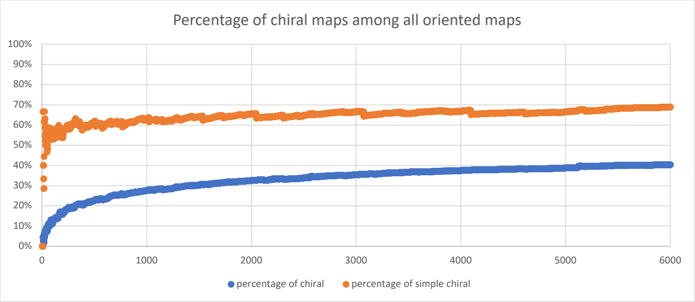
$\mathbb{H}/T$ ... map of type $\{k,m\}$

$\mathrm{Aut}(\mathbb{H}/T) = N_{\Delta^*}(T)/T$

G. Jones : Exists, provided $\mathbb{H}/T$ is hyperbolic.

$\mathbb{H}/L$ is then chiral cover of $\mathbb{H}/T$ of type $\{k,m\}$.

Proportion of chiral – by the number of edges



Conjecture: $$\frac{\#\text{chiral}}{\#\text{rotary orientable}} \longrightarrow 1.$$

HALF-ARC-TRANSITIVITY OF GRAPHS

Half-arc-transitive group actions

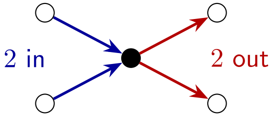
Let $G \leq \mathrm{Aut}(\Gamma)$. Suppose G is vertex- and edge-transitive, but not arc-transitive.

Then we say Γ is $(G, \frac{1}{2})$-arc-transitive.

Also, $(\mathrm{Aut}(\Gamma), \frac{1}{2})$-arc-transitive $=$ $\frac{1}{2}$-arc-transitive.

G has 2 orbits on arcs: an arc and its reverse lie in *different* orbits.

Each orbit induces a digraph $\vec{\Gamma}$ with $G \leq \mathrm{Aut}(\vec{\Gamma})$.



valency 4: orienting the edges yields a G-arc-transitive digraph

Consequently: $(G, \frac{1}{2})$-arc-transitive graphs have even valence.

Half-arc-transitive graphs

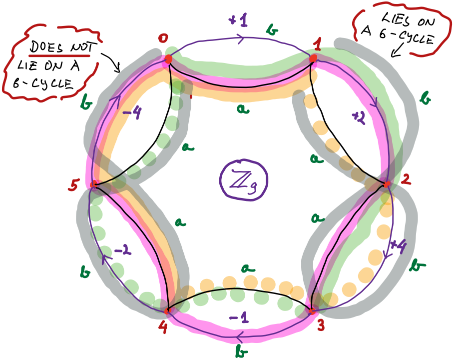
Suppose Γ is $(G, \frac{1}{2})$-arc-transitive. There are two possibilities:

- $\exists$ an automorphism inverting an edge: Γ is arc-transitive (GHAT)
- no such automorphism: Γ is $\frac{1}{2}$-arc-transitive (HAT).

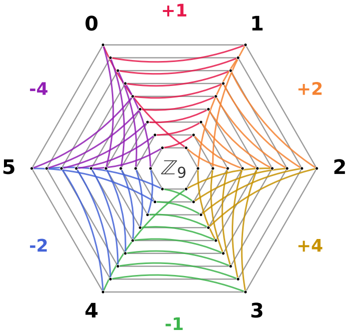
Note:

- An arc-transitive graph might fail to be GHAT.
- Even if it is GHAT, there may be no automorphism swapping the two G-arc-orbits: the induced digraphs might still be non-isomorphic.

Bouwer graph

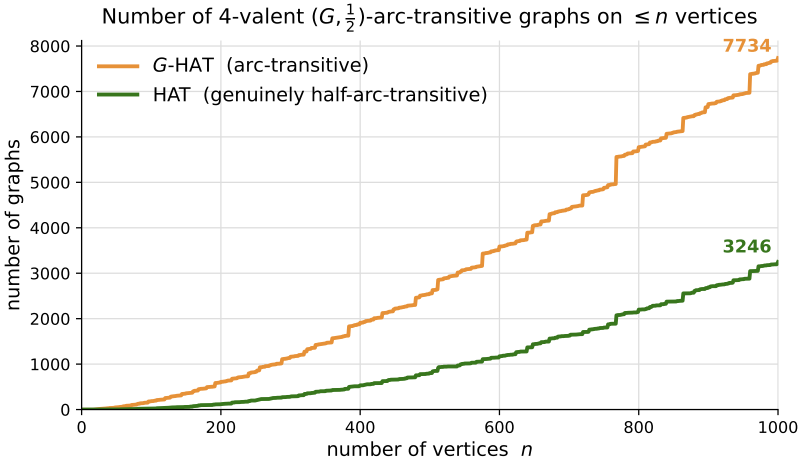


voltage graph on $\mathbb{Z}_9$

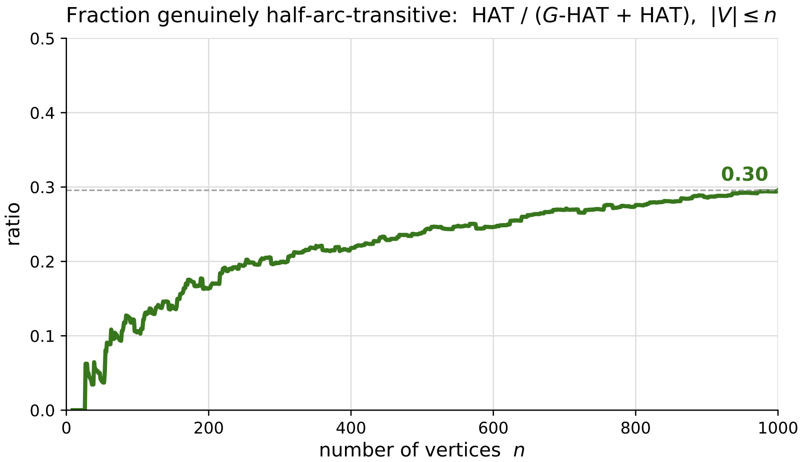


covering graph (54 vertices)

Counting 4-valent $(G, \frac{1}{2})$-arc-transitive graphs



Proportion of HATs among 4-valent $(G, \frac{1}{2})$-arc-transitive



SEMISYMMETRIC GRAPHS

Semisymmetric graphs

Let Γ be a regular graph.

It is semisymmetric if it is edge-transitive but not vertex-transitive.

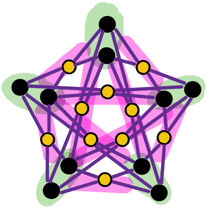
Relative to a subgroup $G \leq \mathrm{Aut}(\Gamma)$:
Γ is G-semisymmetric if G is transitive on edges, not on vertices.

Always bipartite: the two parts are G-vertex-orbits.

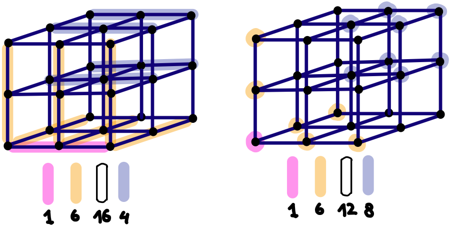
If Γ is G-semisymmetric, then two things can happen:

- Γ is semisymmetric (no edge-reversal) restricted symmetry;
- Γ is arc-transitive (an edge can be reversed).

Two semisymmetric graphs

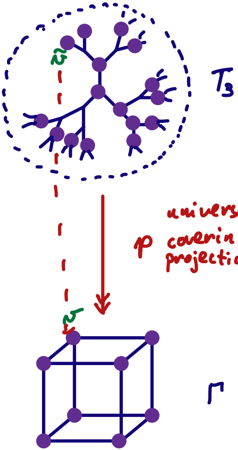


Folkman's graph



Bouwer's configuration → Gray graph

Bass–Serre theory viewpoint



p universal coverin projection

$$\begin{tikzcd}
& & \mathrm{Aut}(T_3) \\
\widetilde{G} \arrow[r, hook] \arrow[d, "p^*|_{\widetilde{G}}"'] & \widetilde{\mathrm{Aut}(\Gamma)} \arrow[ur, hook] \arrow[d, "p^*"] & \\
G \arrow[r, hook] & \mathrm{Aut}(\Gamma) &
\end{tikzcd}$$

locally compact totally disconn.

discrete (since finite stabiliser)

$\Gamma \cong T_3 / \mathrm{Ker}(p^*)$

IMPORTANT:

$\mathrm{Ker}(p^*) \cong \pi_1(\Gamma) \cong F_\beta$ acts freely on T_3

$\widetilde{G}_{\tilde{v}} \xrightarrow{\cong} G_v$

G is v.t, e.t., a.t., ...
$\Longleftrightarrow$
$\widetilde{G}$ is v.t, e.t., a.t., ...

Discrete edge-transitive subgroups of $\mathrm{Aut}(T_r)$

Therefore: Every finite G-edge-transitive r-valent graph arises from a discrete edge-transitive subgroup $\tilde{G} \leq \mathrm{Aut}(T_r)$.

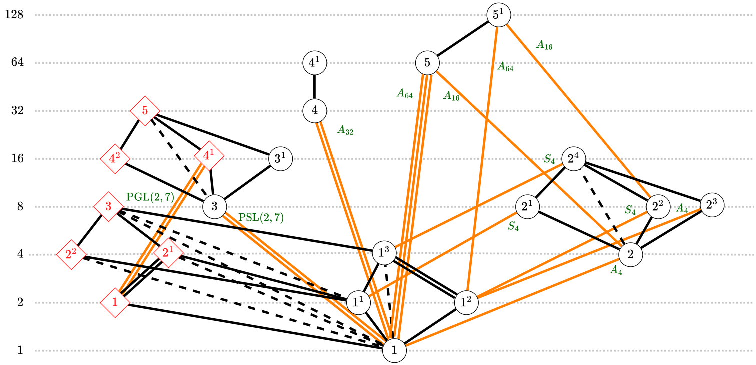
Often there are infinitely many (conjugacy classes of) such subgroups. But for some valencies, there are only finitely many!

Valency 3:

Tutte, Djoković–Miller, Goldschmidt: There are $7 + 15$ (conjugacy classes of) edge-transitive discrete subgroups of $\mathrm{Aut}(T_3)$.

Therefore: Finite cubic edge-transitive graphs are of one of 7 DjM-types or one of 15 Gold-types.

Inclusions between discrete ET subgroups of $\mathrm{Aut}(T_3)$



Realisability of DjM and Gold types

Does each of the 22 DjM and Gold groups have a realisation in finite graphs?

All these groups are finitely generated and residually finite. This assures there exists a finite G-edge-transitive graph for G being of an arbitrary symmetry type.

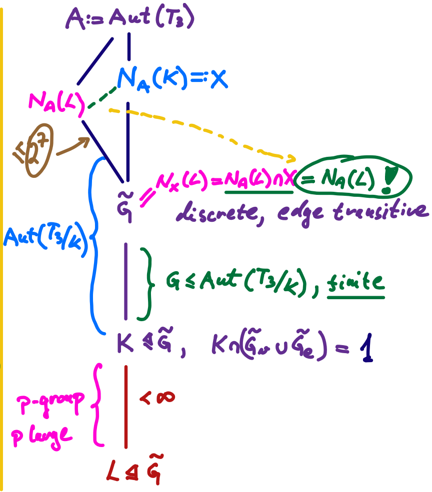
However, can we assure that $G = \mathrm{Aut}(\Gamma)$? This is again a question of restricted symmetry.

The answer is YES:

Conder, PP (2026): There are infinitely many finite connected cubic edge-transitive graphs of each of the 22 DjM and Gold types.

Realisability of DjM and Gold types, cont'd

- We are given $\tilde{G} \le \mathrm{Aut}(T_3)$, discrete and edge-transitive.
- $\tilde{G}$ is residually finite $\to$ $\exists K \trianglelefteq_{<\infty} \tilde{G}$ avoiding $\tilde{G}_v, \tilde{G}_{\tilde{e}}$
- T_3/K is a finite, $\tilde{G}/K \le \mathrm{Aut}(T_3/K)$
- But $\mathrm{Aut}(T_3/K) = N_A(K) =: X$!!
- We want $L \le K$ s.t. $N_A(L) = \tilde{G}$
- We can: $L \le K$ s.t. $N_X(L) = \tilde{G}$!
  also: $[K:L] = p^\alpha$, p arbitrary prime
- By choosing p large, we get K/L char $N_A(L)/L \rightsquigarrow N_A(L) \le X$



Main ingredient – the group lifting theorem

Spiga, PP (2017): Let Γ be a finite graph s.t. $\mathrm{Aut}(\Gamma)$ acts faithfully on $H_1(\Gamma,\mathbb{Z})$, let $G \leq \mathrm{Aut}(\Gamma)$ and let p be an odd prime.

Then there exists a regular covering projection $\wp\colon \tilde{\Gamma} \to \Gamma$ s.t.

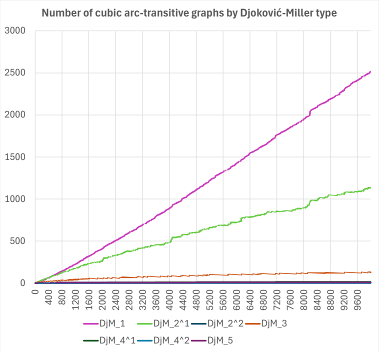
- G is the maximal group that lifts along $\wp$;
- $\mathrm{CT}(\wp)$ is a (finite) p-group.

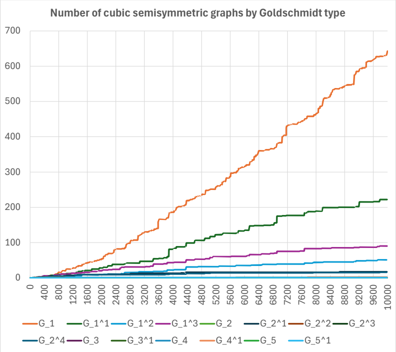
We are not quite happy with this. We would like to add:

- Every automorphism of $\tilde{\Gamma}$ projects to an automorphism of Γ.

We conjecture this is true, but we have no proof!

Enumeration by Gold and DjM type – empirical data





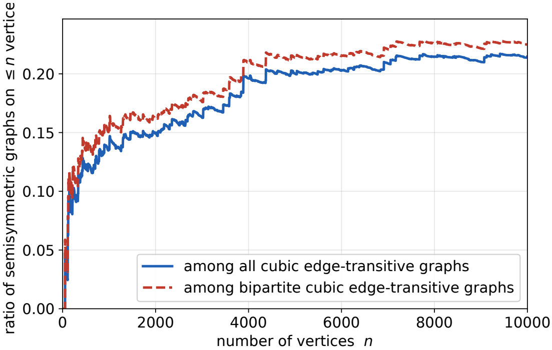
Enumeration by Gold and DjM type – asymptotics

Conder, PP (2026): For every DjM or Gold type T there exist constants a and b, such that the number $f_T(n)$ of all cubic ET graphs of symmetry type T of order at most n satisfies:
$n^{a \log n} - 1 \leq f_T(n) \leq n^{b \log n}$.

We have no knowledge about the constants a and b for different types T. Consequently, we can't deduce anything about the asymptotics of $f_{T_1}(n)/f_{T_2}(n)$ for different types T_i.

The question whether semisymmetry in cubic graphs dominates arc-transitivity thus remains open!

Proportion of cubic semisymmetric graphs

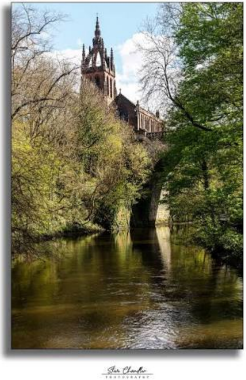


Proportion of semisymmetric among all bipartite cubic edge-transitive graphs.

Conclusion

- Symmetry is beautiful, but a little bit of asymmetry is more interesting.
- Proving non-existence of symmetry is often more difficult than existence.
- Typical empirical data: Small objects with restricted symmetry are rare, but they eventually become prevalent.
- Even if true, proving this is very difficult.

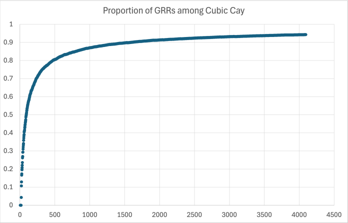
And now I have sadly taxed your patience:
and I fear I have exhausted it and not exhausted my subject!



BONUS MATERIAL

Proportion of GRRs

A Cayley graph $\mathrm{Cay}(G, S)$ is a GRR, provided that $\mathrm{Aut}(\Gamma) \cong G$.



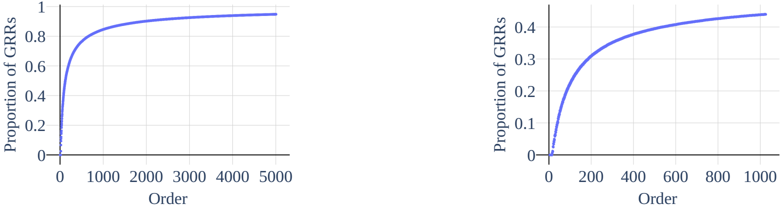
The data suggest: $\frac{\mathrm{GRR}_3(n)}{\mathrm{Cay}_3(n)} \to 1$ as $n \to \infty$.

Recently proved by S. Zhang and B. Xia:

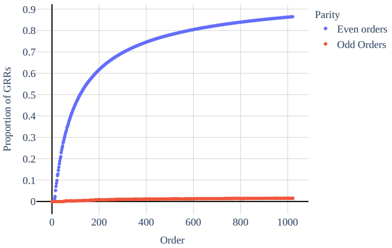
The proportion of GRRs among all Cayley graphs tends to 1.

Unfortunately, this says nothing about Cayley graphs of fixed valence.

Proportion of GRRs – valence 3 and 4

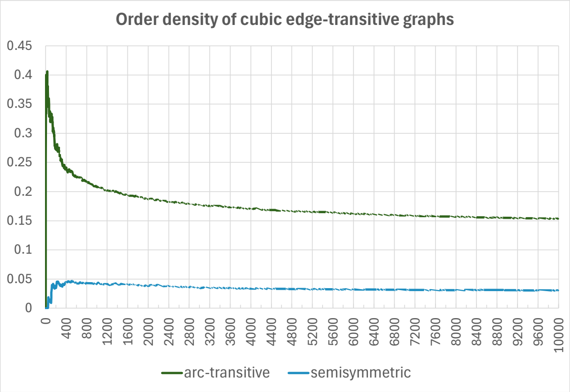


% of cubic (L) and quartic (R) GRRs.



% of quartic GRRs for odd and even orders.

Back to cubic edge-transitive: Order density



Conder, Verret, Young (2024): The natural density of orders of cubic edge-transitive graphs is 0.

Semisymmetric graphs – some history

Folkman (1967): Introduced the concept, found the smallest on 20 vertices, proved there are none of order $2p$ or $2p^2$.

Gray (1932): Found a cubic semisymmetric graph on 54 vertices, recognised as semisymmetric by Bouwer (1968).

Conder, Malnič, Marušič, PP (2006): All cubic semisymmetric graphs of order ≤ 768 determined; extended by Conder and PP (2018) to $10{,}000$;

`https://fostercensus.graphsym.net`

Wilson, PP (2004–): Started collecting 4-valent semisymmetric graphs of order ≤ 512. Wilson maintains a beautiful census:

`https://jan.ucc.nau.edu/~swilson/C4FullSite/index.html`

Conder, Verret (2019): All of order ≤ 63 regardless of the valence:

`www.math.auckland.ac.nz/~conder/`