Controlling the automorphism group of a covering graph

Primož Potočnik

JOINT WORK WITH

Pablo Spiga

Department of Mathematics
Faculty of Mathematics and Physics
University of Ljubljana

International Workshop on Symmetries of Graphs and Networks
Sanya, January 30th, 2018

Motivation, part 1

- Let Γ be a finite connected cubic G-arc-transitive graph. Then G is of one of 7 "types":
  - G is 1-arc-regular;
  - G is 2-arc-regular (two "types");
  - G is 3-arc-regular;
  - G is 4-arc-regular (two "types");
  - G is 5-arc-regular.
- It is easy to construct pairs (Γ, G) for each of the above possibilities.
- Problem (Djoković and Miller, 1980): Can this be achieved with $G = \mathrm{Aut}(\Gamma)$?

Motivation, part 2

- Let Γ be a finite connected tetravalent G-half-arc-transitive graph. Then (by Marušič and Nedela):
  - $|G_v| = 2^s$ for some $s \geq 1$;
  - for every s, there is a finite number of "types" for G;
- Easy to find pairs (Γ, G) for each of the above types.
- Marušič, Nedela, 2001: Can this be achieved with $G = \mathrm{Aut}(\Gamma)$?
- Yes, for some types, unknown in general!

Possible general approach to such problems

General problem: We are given a pair (Γ, G) of a given "type", but such that $G < \mathrm{Aut}(\Gamma)$. Can we find another pair $(\tilde{\Gamma}, \tilde{G})$ of the same "type", where $\tilde{G} = \mathrm{Aut}(\tilde{\Gamma})$.

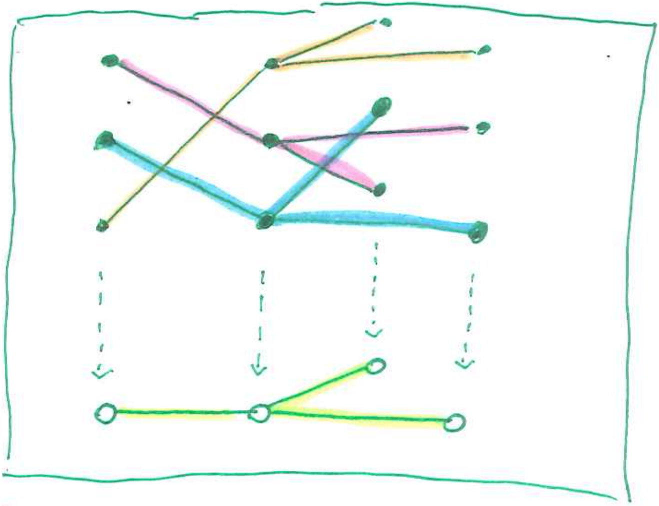
Covering projections, part I

Let $\tilde{\Gamma}$ and Γ be connected graphs.

A graph morphism $\wp \colon \tilde{\Gamma} \to \Gamma$ is a covering projection provided that

- $\wp$ is a surjection (epimorphism);
- for every $v \in V_{\tilde{\Gamma}}$ the restriction $\wp_v \colon \tilde{\Gamma}(v) \to \Gamma(\wp(v))$ is a bijection.

Covering projections, local situation



Fibres and induced automorphisms

Let $\wp\colon \tilde{\Gamma} \to \Gamma$ be a covering projection.

- For a *vertex* or *dart* x of Γ, the preimage $\wp^{-1}(x)$ is called a fibre of x (we have vertex-fibres and dart-fibres).
- An automorphism $\tilde{g} \in \mathrm{Aut}(\Gamma)$ that maps fibres to fibres induces an automorphism g of Γ.
- In this case we say: $\tilde{g}$ projects, g lifts, and $\tilde{g}$ is a lift of g.
- Let $G \leq \mathrm{Aut}(\Gamma)$. If every $g \in G$ lifts, then G lifts. The set $\tilde{G}$ of all lifts of all $g \in G$ is a group, called the lift of G.
- The lift of the trivial group $\langle \mathrm{id}_\Gamma \rangle \leq \mathrm{Aut}(\Gamma)$ is called the group of covering tranformations ... $\mathrm{CT}(\wp)$.

Regular covers and its nice feature

If $\mathrm{CT}(\wp)$ is transitive on every fibre, then $\wp$ is a regular covering projection.

Let $\wp\colon \tilde{\Gamma} \to \Gamma$ be a regular covering projection. Suppose that $G \leq \mathrm{Aut}(\Gamma)$ lifts to $\tilde{G}$. Then:

- G is vertex-transitive iff $\tilde{G}$ is vertex-transitive;
- G is edge-transitive iff $\tilde{G}$ is edge-transitive;
- G is s-arc–transitive iff $\tilde{G}$ is s-arc-transitive;
- if $v = \wp(\tilde{v})$, then $\tilde{G}_{\tilde{v}} \cong G_v$ and $\tilde{G}_{\tilde{v}}^{\tilde{\Gamma}(\tilde{v})} \cong G_v^{\Gamma(v)}$.

In this sense regular covering projections preserve "type".

The problem

Recall our problem: For a (Γ, G) of a given "type", find another pair $(\tilde{\Gamma}, \tilde{G})$ of the same "type" satisfying $\tilde{G} = \mathrm{Aut}(\tilde{\Gamma})$.

We can now solve this by:

finding a regular covering projection $\wp\colon \tilde{\Gamma} \to \Gamma$ s.t.:

1. G lifts along $\wp$, but no larger group does;
2. Every automorphism of $\mathrm{Aut}(\tilde{\Gamma})$ projects to some automorphism of Γ.

This works since "type" is preserved by $\wp$.

A word of warning

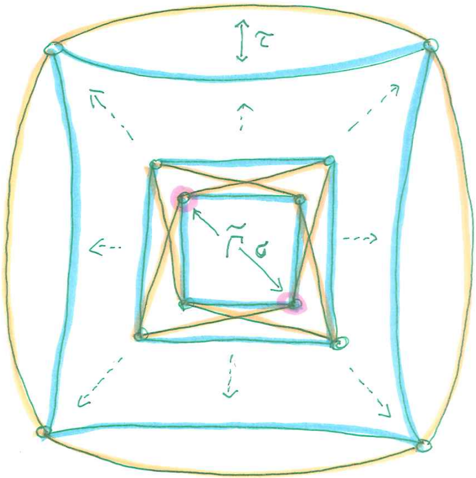
In general, it is difficult to control the automorphism group of $\tilde{\Gamma}$.

If $\wp\colon \tilde{\Gamma} \to \Gamma$ is a regular covering projection, then it may happen that:

(1) There are automorphisms of Γ that do not lift;

(2) There are automorphisms of $\tilde{\Gamma}$ that do not project.

A word of warning, example



$\tilde{\Gamma} \cong K_{4,4}$;

σ does not project;

τ has no lift.

Main result

Theorem (P. Spiga, PP, 2017)

Let Γ be a finite graph s.t. $\mathrm{Aut}(\Gamma)$ acts faithfully on the integral cycle space $H_1(\Gamma, \mathbb{Z})$, let $G \leq \mathrm{Aut}(\Gamma)$ and let p be an odd prime.

Then there exists a regular covering projection $\wp \colon \tilde{\Gamma} \to \Gamma$ s.t.

- *G is the maximal group that lifts along $\wp$;*
- *$\mathrm{CT}(\wp)$ is a (finite) p-group.*

We are not quite happy with this. We would like to add:

- Every automorphism of $\tilde{\Gamma}$ projects to an automorphism of Γ.

We conjecture this is true, but we have no proof!

Group theoretical reformulation

Theorem

Let p be a prime, let T be an infinite tree, let $G \leq \mathrm{Aut}(T)$, let N be a non-identity normal subgroup of G of finite index such that $N_x = 1$ for every vertex and for every edge x of T, and let $H = \mathbf{N}_{\mathrm{Aut}(T)}(N)$. If H/N acts faithfully on the integral cycle space of T/N, then there exists a normal subgroup P of N of finite index such that $\mathbf{N}_H(P) = G$ and that N/P is p-group.

In order to prove the conjecture, we would need to have $\mathbf{N}_{\mathrm{Aut}(T)}(P) = G$ and not just $\mathbf{N}_H(P) = G$

Some consequences: cubic arc-transitive

Theorem

Let Γ be a finite cubic G-arc-transitive graph. Then there exists a regular covering projection $\wp\colon \tilde{\Gamma} \to \Gamma$ (with $\tilde{\Gamma}$ finite) such that $\mathrm{Aut}(\tilde{\Gamma})$ is the lift of G.

Some consequences, proof

- Let Γ be a finite cubic G-arc-transitive graph. It is known that $|G_v|$ divides 48. It can be seen that $\mathrm{Aut}(\Gamma)$ acts faithfully on $H_1(\Gamma, \mathbb{Z})$.
- Choose $p > 16|G|$.
- By The Theorem, there exists $\wp\colon \tilde{\Gamma} \to \Gamma$ such that G is the maximal group that lifts. Also, $P := \mathrm{CT}(\wp)$ is a p-group.
- Let $\tilde{G}$ be the lift of G and let $\tilde{A} = \mathrm{Aut}(\tilde{\Gamma})$. Then $|\tilde{A} : \tilde{G}| \leq 16$.
- Therefore $|\tilde{A}| \leq 16|\tilde{G}| = 16\,|P|\,|G| \leq |P|^2$.
- Hence P is a normal Sylow p-subgroup of $|\tilde{A}|$. In particular, $\tilde{A}$ projects.
- Hence $\tilde{G} = \tilde{A}$.

Some consequences, 2-arc-transitive

- What is special about cubic arc-transitive graphs?
- Answer: The bound on the vertex-stabiliser.
- The same argument applies for any such situation, for example for 2-arc-transitive graphs of any valence, or for arc-transitive for odd prime valence.

Theorem

Let Γ be a finite $(G, 2)$-arc-transitive graph (or G-arc-transitive of prime valence). Then there exists a regular covering projection $\wp\colon \tilde{\Gamma} \to \Gamma$ (with $\tilde{\Gamma}$ finite) such that $\mathrm{Aut}(\tilde{\Gamma})$ is the lift of G.

... and several other similar theorems...

Alas

Our theorem is not good enough to solve the problem of Marušič and Nedela:

Does there exist a tetravalent half-arc-transitive graph of every possible "type" (in particular, with arbitrary large non-abelian vertex-stabiliser).

But if "conjecture" is true, then the answer to the above is affirmative.