Tetravalent edge-transitive graphs

Let Γ be a (finite, connected) edge transitive 4-regular graph.
Then one of the following happens:

- Γ is arc-transitive;
- Γ is vertex-transitive, not arc-transitive; $\frac{1}{2}$-arc-transitive
- Γ is not vertex-transitive. semisymmetric

Census

Steve and I started compiling on a census of 4-valent edge-transitive graphs back in 2004. Steve built a webpage:

`https://jan.ucc.nau.edu/~swilson/C4FullSite/index.html`

Census complete for:

- arc-transitive graphs, up to order 640, PSV (2013);
- $\frac{1}{2}$-arc-transitive graphs, up to 1000 vertices, PSV (2015);

But incomplete for (some types of) semisymmetric graphs.

The most up-to-date list of 4-valent semisymmetric graphs (up to 512 vertices) that we know of can be found here:

`https://graphsym.net`

Constructing 4-valent semisymmetric graphs

TASK: Find new 4-valent semisymmatric graphs, not yet known.

Two approaches:

Clever constructions of infinite families; see:

https://adam-journal.eu/index.php/ADAM/article/view/1269

Using amalgams: $\quad G_v \hookleftarrow G_{uv} \hookrightarrow G_u, \quad G = \langle G_v, G_u \rangle.$

- G_v and G_u are finite groups;
- $[G_v : G_{uv}] = 4,\ [G_u : G_{uv}] = 4;$
- If $K \leq G_{uv},\ K \lhd G_u,\ K \lhd G_v,$ then $K = 1;$
- I.e.: (G_v, G_{uv}, G_u) is a finite faithful amalgam of index $(4, 4)$;

Reconstruction from an amalgam

Take a finite faithful amalgam of index $(4,4)$: $A \hookleftarrow B \hookrightarrow C$

Construct $U := A *_B C$ as a f.p. group;

Find all $N \trianglelefteq U$, $[U : N] \leq 256 \cdot |A|$.

For each such N, build the quotient $G := U/N$ and construct Γ:

$$\mathrm{V}(\Gamma) = G/A \cup G/C, \quad \mathrm{E}(\Gamma) = \{\{Ag, Cg\} : g \in G\}.$$

Check if Γ is 4-valent and not vertex-transitive;

Finite faithful amalgams of index $(4, 4)$

Infinitely many! We can group them up to the local type (L_A, L_C):

$$L_A \cong A/\mathrm{core}_A(B)\,(\cong G_v^{\Gamma(v)}), \quad L_C \cong C/\mathrm{core}_C(B)\,(\cong G_u^{\Gamma(u)}).$$

Note that: $L_A, L_C \cong C_4, V_4, D_4, A_4, S_4$;

What is already done:

- $L_A, L_C \in \{C_4, V_4\}$ (then necessarily $B = 1$);
- $L_A, L_C \in \{A_4, S_4\}$ (Morgan, PP (20??));
- Some other amalgams with small B;

What still needs to be done:

- Amalgams where $L_A \in \{C_4, V_4\}$ and $L_C \in \{A_4, S_4\}$;
- Amalgams where one of L_A, L_C is D_4;

TASK: Find as many amalgams as possible and construct graphs.