Generating and cataloging symmetric graphs

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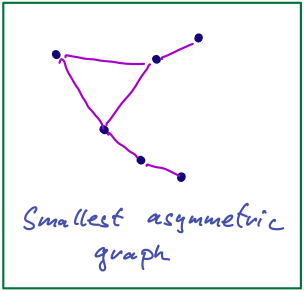
Symmetry

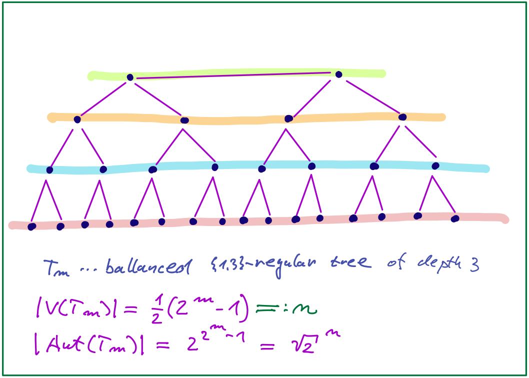
- Symmetry of a graph: permutation of V that preserves $\sim$.
- Almost all graphs have trivial automorphism group.



- Out of the rest, almost all have exactly one nontrivial automorphism!
- Graphs with symmetries are rare!

Graphs with many automorphisms

But some graphs have a huge number of automorphisms.



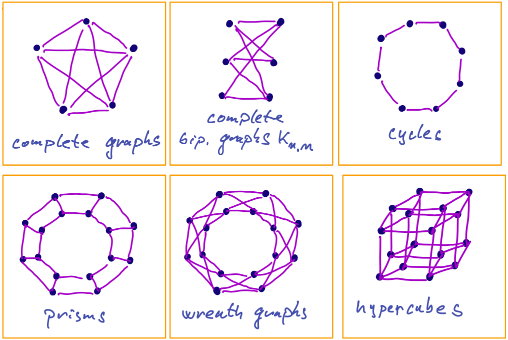
Downside of this example: $\mathrm{Aut}(T_m)$ has $m + 1$ *orbits* on vertices.

I.e., not all vertices are *equivalent*.

Vertex transitive graphs

Measure of symmetry: Not the number of automorphisms,
but rather number of orbits.

Definition: Γ is vertex-transitive if $\mathrm{Aut}(\Gamma)$ has a single orbit on $V(\Gamma)$.



And many more ...

Interesting properties of vertex-transitive graphs

- Vertex-connectivity $\geq \frac{2\times(\text{valence }+1)}{3}$;
- Edge-connectivity $=$ valence;
- There is a matching that misses at most one vertex.
- every edge is contained in a maximal matching.
- Lovasz' conjecture: Every connected Cayley graph, except K_2, is hamiltonian. Every connected vertex-transitive graph, except five known exceptions, is hamiltonian.
- Vertex-transitive snarks: The only known vertex-transitive snarks are the Petersen graph and its truncation.

Higher types of symmetry

In vertex-transitive graphs, all vertices are equivalent. But more can hold:

- $\mathrm{Aut}(\Gamma)$ can be transitive on arcs (ordered pairs of adjacent vertices): arc-transitive graphs.
- $\mathrm{Aut}(\Gamma)$ can be transitive on s-arcs (reduced walks of length s): s-arc-transitive graphs.
- For example, cycles are s-arc-transitive for every s. The Petersen graph is 3-arc-transitive. The Tutte's 8-cage (3-regular graph of girth 8 on 30 vertices) is 5-arc-transitive. The incidence graph of a generalised hexagon (of valence 4 and order 728) is 7-arc-transitive.
- There are no 8-arc-transitive graphs of valence ≥ 3 (Weiss, 1980) and no 6-arc-transitive cubic graphs (Tutte, 1947).

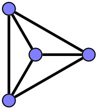
MAIN MESSAGE

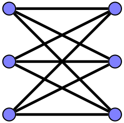
Symmetric graphs are rare,

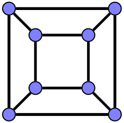
but very much worth investigating.

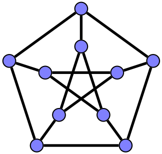
Construction of catalogues of symmetric graph

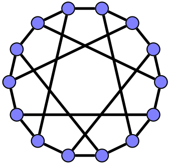
How to construct and catalogue symmetric graphs.

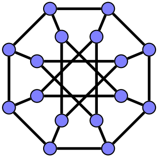

4, K_4

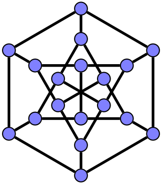

6, $K_{3,3}$

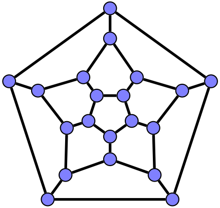

8, Q_3

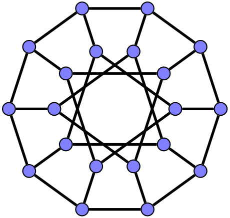

10, Petersen graph

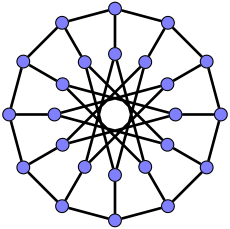

14, Heawood


16, Möbius-Kantor


18, Pappus


20, Dodecahedron

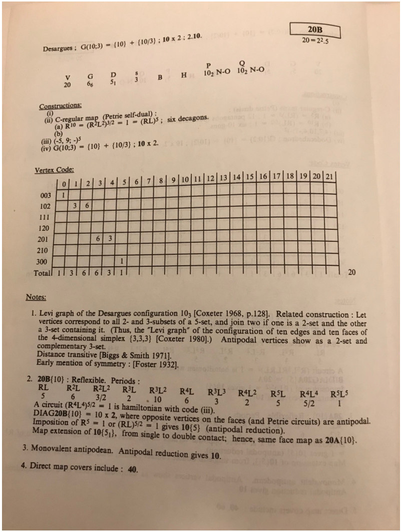

20, Desargues


24, Nauru

Foster's census of cubic arc-transitive graphs

- Foster started collecting the graphs in 1930s.
- First presented at the "Conference on Graph Theory and Combinatorial Analysis, Waterloo, 1966".
- In 1988, book: up to order 512 (only a few were missing).
- Each graph had its own page in the book, with construction, several parameters, relationship with other graphs etc.

Foster census



Desargues ; G(10;3) = {10} + {10/3} ; **10 x 2 : 2.10.**

20B

$20 = 2^2.5$

V	G	D	s	B	H	P	Q
20	6_6	5_1	3			10_2 N-O	10_2 N-O

Constructions:

(i)

(ii) C-regular map (Petrie self-dual) :

(a) $R^{10} = (R^2L^2)^{3/2} = 1 = (RL)^5$; six decagons.

(b)

(iii) $(-5, 9; -)^5$

(iv) G(10;3) = {10} + {10/3} ; **10 x 2.**

Vertex Code:

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	
003	1																						
102		3	6																				
111																							
120																							
201				6	3																		
210																							
300						1																	
Total	1	3	6	6	3	1																	20

Notes:

1. Levi graph of the Desargues configuration 10_3 [Coxeter 1968, p.128]. Related construction : Let vertices correspond to all 2- and 3-subsets of a 5-set, and join two if one is a 2-set and the other a 3-set containing it. (Thus, the "Levi graph" of the configuration of ten edges and ten faces of the 4-dimensional simplex {3,3,3} [Coxeter 1980].) Antipodal vertices show as a 2-set and complementary 3-set.
Distance transitive [Biggs & Smith 1971].
Early mention of symmetry : [Foster 1932].

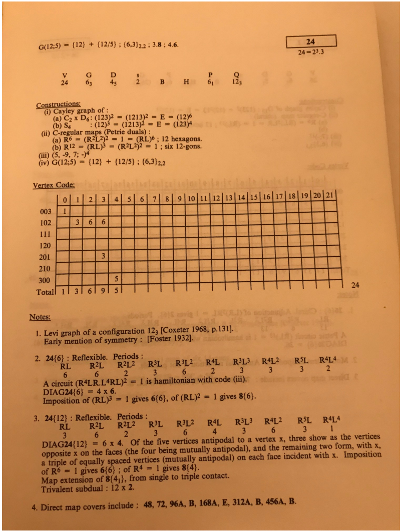
2. **20B**{10} : Reflexible. Periods :

RL	R^2L	R^2L^2	R^3L	R^3L^2	R^4L	R^3L^3	R^4L^2	R^5L	R^4L^4	R^5L^5
5	6	3/2	2	10	6	3	2	5	5/2	1

A circuit $(R^4L^4)^{5/2} = 1$ is hamiltonian with code (iii).
DIAG**20B**{10} = 10 x **2**, where opposite vertices on the faces (and Petrie circuits) are antipodal.
Imposition of $R^5 = 1$ or $(RL)^{5/2} = 1$ gives **10**{5} (antipodal reduction).
Map extension of **10**{5_1}, from single to double contact; hence, same face map as **20A**{10}.

3. Monovalent antipodean. Antipodal reduction gives **10**.

4. Direct map covers include : **40**.



G(12;5) = {12} + {12/5} ; $\{6,3\}_{2,2}$; **3.8** ; **4.6.**

24

$24 = 2^3.3$

V	G	D	s	B	H	P	Q
24	6_3	4_5	2			6_1	12_3

Constructions:

(i) Cayley graph of :

(a) $C_2 \times D_6$: $(123)^2 = (1213)^2 = E = (12)^6$

(b) S_4 : $(12)^3 = (1213)^2 = E = (123)^4$

(ii) C-regular maps (Petrie duals) :

(a) $R^6 = (R^2L^2)^2 = 1 = (RL)^6$; 12 hexagons.

(b) $R^{12} = (RL)^3 = (R^2L^2)^2 = 1$; six 12-gons.

(iii) $(5, -9, 7; -)^4$

(iv) G(12;5) = {12} + {12/5} ; $\{6,3\}_{2,2}$

Vertex Code:

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	
003	1																						
102		3	6	6																			
111																							
120																							
201				3																			
210																							
300					5																		
Total	1	3	6	9	5																		24

Notes:

1. Levi graph of a configuration 12_3 [Coxeter 1968, p.131].
Early mention of symmetry : [Foster 1932].

2. **24**{6} : Reflexible. Periods :

RL	R^2L	R^2L^2	R^3L	R^3L^2	R^4L	R^3L^3	R^4L^2	R^5L	R^4L^4
6	6	2	3	6	2	3	3	3	2

A circuit $(R^4LR.L^4RL)^2 = 1$ is hamiltonian with code (iii).
DIAG**24**{6} = 4 x **6.**
Imposition of $(RL)^3 = 1$ gives **6**{6}, of $(RL)^2 = 1$ gives **8**{6}.

3. **24**{12} : Reflexible. Periods :

RL	R^2L	R^2L^2	R^3L	R^3L^2	R^4L	R^3L^3	R^4L^2	R^5L	R^4L^4
3	6	2	3	6	4	3	6	3	1

DIAG**24**{12} = 6 x **4.** Of the five vertices antipodal to a vertex x, three show as the vertices opposite x on the faces (the four being mutually antipodal), and the remaining two form, with x, a triple of equally spaced vertices (mutually antipodal) on each face incident with x. Imposition of $R^6 = 1$ gives **6**{6} ; of $R^4 = 1$ gives **8**{4}.
Map extension of **8**{4_1}, from single to triple contact.
Trivalent subdual : 12 x **2.**

4. Direct map covers include : **48, 72, 96A, B, 168A, E, 312A, B, 456A, B.**

Foster's census of cubic arc-transitive graphs

- First complete (computer generated) version (up to 768 vertices) was obtained in 2001.
  (Conder, Dobcsányi)
- The census is now extended up to 10 000 vertices (3 815 graphs).
  (Conder)
- Foster did not guarantee completeness (he missed a few graphs), Conder's census is complete.
- Incomplete censuses can be found by clever constructions . To guarantee completeness, we need some sort of exhaustive search.
- Symmetric graphs, are typically found via their automorphism groups.

How was the census computed?

Γ ... connected k-valent graph, $G \le \mathrm{Aut}(\Gamma)$ arc-transitive.

- Let $\wp\colon \mathcal{T}_k \to \Gamma$ be the universal covering projection.
- Universality condition: The group G 'lifts' to some arc-transitive $\tilde{G} \le \mathrm{Aut}(\Gamma)$. In fact, $\tilde{G} \cong G_v *_{G_{uv}} G_{\{u,v\}}$.
- Important: $G_v \cong \tilde{G}_{\tilde{v}}$. In particular, $\tilde{G}_{\tilde{v}}$ is finite.

  That is, $\tilde{G}$ is a discrete arc-transitive subgroup of $\mathcal{T}_k$.
- Group of covering transformations: $N \trianglelefteq \tilde{G}$, $N \cap \tilde{G}_{\tilde{v}} = N \cap \tilde{G}_{\{\tilde{v},\tilde{u}\}} = 1$, transitive on each fibre of $\wp$.
- Consequently: $\Gamma \cong \mathcal{T}/N$, $G \cong \tilde{G}/N$.
- Moreover, $G_v \cong \tilde{G}_{\tilde{v}}N/N$ and $G_{\{u,v\}} \cong \tilde{G}_{\{\tilde{v},\tilde{u}\}}N/N$.
- Therefore: $\Gamma \cong \mathrm{Cos}(\tilde{G}/N, \tilde{G}_{\tilde{v}}N/N, \tilde{G}_{\{\tilde{v},\tilde{u}\}}N/N)$.

How was the census computed?

Say we want to find all connected k-valent graphs of order $\leq M$ admitting an arc-transitive group G with $|G_v| \leq m$ (for fixed k, m and M).

Algorithm:

- Find all arc-transitive discrete groups $\tilde{G} \leq \mathrm{Aut}(\mathcal{T}_k)$ with $|\tilde{G}_{\tilde{v}}| \leq m$. (There are only finitely many and can be effectively found.)
- For each such $\tilde{G}$, find all $N \trianglelefteq \tilde{G}$ with $|\tilde{G} : N| \leq M\,|\tilde{G}_{\tilde{v}}|$.
- For each such $\tilde{G}$ and N, construct $\Gamma = \mathrm{Cos}(\tilde{G}/N, \tilde{G}_{\tilde{v}}N/N, \tilde{G}_{\{\tilde{v},\tilde{u}\}}N/N)$, and test if it is k-valent.
- Reduce modulo graph isomorphism.

Demonstration – cubic case

The algorithm only finds graphs admitting arc-transitive groups of bounded vertex-stabiliser. For cubic graphs, this restriction is not needed:

Theorem (Tutte, 1947)

If Γ is a connected cubic arc-transitive graph, then $|\mathrm{Aut}(\Gamma)_v| \le 48$.

Corollary: There is a finite number of conjugacy classes of discrete arc-transitive subgroups of $\mathrm{Aut}(\mathcal{T}_3)$. In fact, there are 7 such classes. The representatives were determined by Djoković and Miller in 1980.

SWITCH TO MAGMA DEMONSTRATION

Other complete catalogues

Tutte's result can be generalised to some other symmetry types :

- Cubic semisymmetric graphs, up to 10,000 vertices; 1,043
- 4-valent arc-transitive graphs, up to 640 vertices; 4,820
- Cubic vertex-transitive graphs, up to 1,280 vertices; 111,360
- 4-valent half-arc-transitive graphs, up to 1,000 vertices; 3,246
- 2-valent arc-transitive digraphs on up to 1,000 vertices; 26,457
- Could do: 5-valent edge-transitive graphs up to $\approx$ 4,000 vertices.

All this and more can be found here: https://graphsym.net

Arbitrary valence

All these catalogues were for fixed valence.

Catalogues of symmetric graph of arbitray valence are difficult to construct.

- Royle, Holt, 2022: Census of all vertex-transitive graphs of order up to 48 ($1{,}538{,}868{,}366$ graphs of order 48 only).
- Conder,Verret, 2019: Census of all edge-transitive graphs of order up to 63.

Both these catalogues rely on the determination of all transitive permutation groups on at most 48 points.

Difficulties

Sometimes, we don't know how to bound the order of the group:

- 4-valent smisymmetric graphs (not even up to 100 vertices).

Sometimes, the issue is a vast number of graphs.

Consider 3-valent Cayley graphs on n vertices.

- Each such graph is determined by a group G of order n, and a generating set S of size at most 3.
- There is a vast number of groups generated by S as above, and even more generating sets S.
- Up to order 4094, there are over $1{,}221{,}573$ non-isomorphic 3-valent Cayley graphs.

Main message

- To construct incomplete catalogues: Clever constructions.
- To construct complete catalogues:

  We need to determine possible automorphism groups.

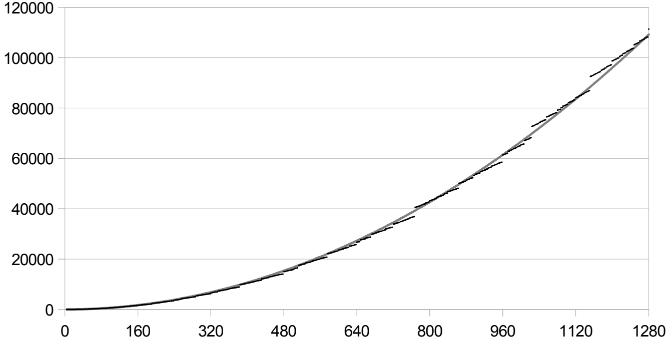
  For that we need to control the size and/or the structure of the group.

Enumeration

Symmetric graphs are rare.

How rare?

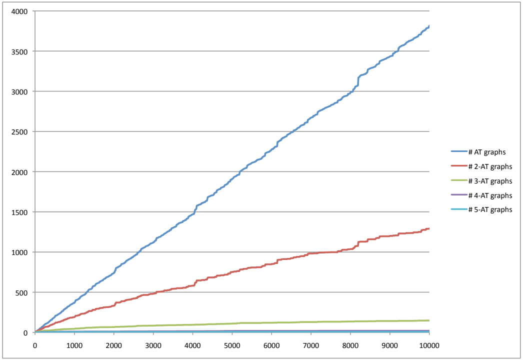
Number of cubic vertex-transitive graphs of order up to n



In gray is the graph of the function $n \mapsto n^2/15$.

Does the number of cubic arc-transitive graphs of order up to n grows as a quadratic function of n?

Number of cubic arc-transitive graphs of order up to n



Does the number of cubic vertex-transitive graphs of order up to n grows as a linear function of n?

Surprise

Theorem (Spiga, PP + Verret)

Let $\mathcal{C}$ be any of the following classes of connected graphs:

- *cubic vertex-transitive;*
- *cubic arc-transitive;*
- *cubic arc-transitive of any fixed Djoković-Miller type;*
- *4-valent arc-transitive;*
- *2-arc-transitive of any fixed valence;*
- ...

Let $f(n) = |\{\Gamma \in \mathcal{C} : |V(\Gamma)| \leq n\}|$. Then there exist positive constants a and b such that

$$n^{a \log n} \leq f(n) \leq n^{b \log n} \quad (\textit{i.e. } f(n) \approx n^{\log n})$$

for all sufficiently large n.

Ingredients of the proof

- [Spiga, PP]: If Γ is a graph admitting $G \leq \mathrm{Aut}(\Gamma)$ (satisfying very mild condition) and p is an odd prime, then there exists a regular covering projection $\wp\colon \tilde{\Gamma} \to \Gamma$ with fibres of p-power size, such that the maximal group that lifts is G.
- [Bass-Serre Theory]: If $\tilde{G} \leq \mathrm{Aut}(\mathcal{T}_d)$ and $N \triangleleft \tilde{G}, N \cap (\tilde{G}_v \cup \tilde{G}_{\{u,v\}}) = 1$, $|\tilde{G} : N| < \infty$, then N is a free group of finite rank.
- [Müller, J.-C. Schlage-Puchta]: Let $\tilde{G}$, let $N \triangleleft \tilde{G}$ s.t. $|\tilde{G} : N| < \infty$, N free of finite rank, let p be a prime, and let $f(n)$ be the number of subgroups of N of p-power index that are normal in $\tilde{G}$ and such that $|\tilde{G}/N| \leq n$. Then $f(n) \approx n^{\log n}$.
- [Various authors]: In each of the classes $\mathcal{C}$ from the theorem, there exists either a constant bound on $|\mathrm{Aut}(\Gamma)_v|$, or at least a very tame bound in terms of $|V(\Gamma)|$.

Comments

- The ideas for this proof come from a classical result of Mann:

  The number of d-generated groups of order p^m is at least $p^{c(d)m^2}$.

- Similar approach proves a number of other enumeration results:
  - Number of regular maps of genus g is asymptotically $\approx g^{\log g}$.
  - For any pair (p, q) such that $\frac{1}{p} + \frac{1}{q} < \frac{1}{2}$, the number of regular maps of type (p, q) and number of edges $\leq n$ is asymptotically $\approx n^{\log n}$.

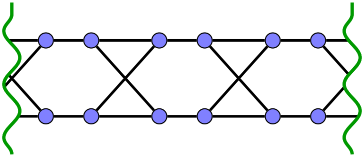
How can catalogues be used—examples

Fixity of graphs:

- $\mathrm{Fix}(\Gamma) =$ "largest number of fixed vertices of $g \in \mathrm{Aut}(\Gamma) \setminus \{1\}$"
- $|V(\Gamma)| - \mathrm{Fix}(\Gamma) =$ "minimal degree of $\mathrm{Aut}(\Gamma)$."
- $\mathrm{RelFix}(\Gamma) = \frac{\mathrm{Fix}(\Gamma)}{|V(\Gamma)|}$.
- Qustion: How large can $\mathrm{RelFix}(\Gamma)$? In particular, for cubic vertex-transitive graphs.

Fixiity of cubic vertex-transitive graphs

Some cubic vertex-transitive graphs have very large fixity:



Split wreath graph SW_m : $\mathrm{Fix}(\mathrm{SW}_m) = |V| - 4$, $\mathrm{RelFix}(\mathrm{SW}_m) \to 1$.

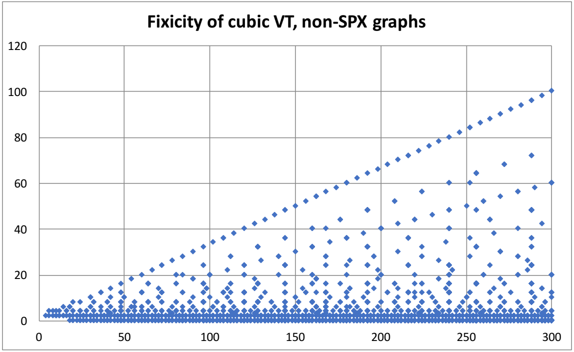
More generally, split Praeger-Xu graphs $\mathrm{SPX}(n, k)$ satisfy

$$\mathrm{Fix}(\mathrm{SPX}(m, k)) = n - k2^{k+1}$$

In particular, for every fixed $k \geq 1$:

$$\mathrm{RelFix}(\mathrm{SPX}(m, k)) \to 1 \text{ as } |V| \to \infty.$$

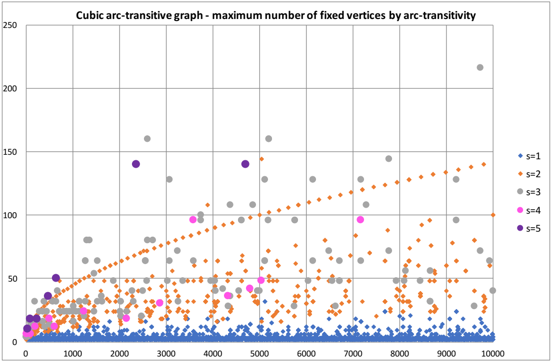
Fixity of cubic vertex-transitive graphs



Theorem (Spiga, PP; 2021)

If Γ is a finite connected cubic vertex-transitive graph, then either it is isomorphic to an SPX-graph or $\mathrm{RelFix}(\Gamma) \leq \frac{1}{3}$.

Fixicity of cubic arc-transitive graphs



Theorem (Spiga, Lehner, PP; 2021)

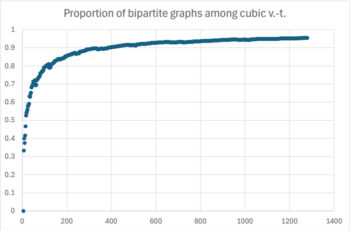
For connected cubic arc-transitive graphs Γ*:*

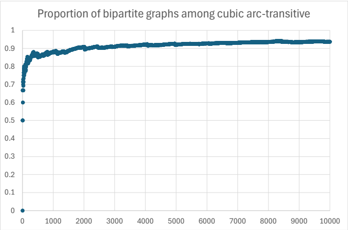
$$\mathrm{RelFix}(\Gamma) \to 0 \quad \textit{as} \quad |V(\Gamma)| \to \infty.$$

Prevalence of bipartness

Take an arbitrary cubic vertex-transitive graph of order up to n.

What is the probability it is bipartite?

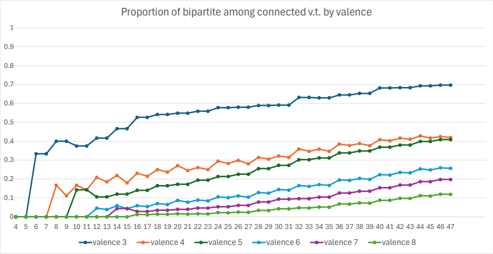


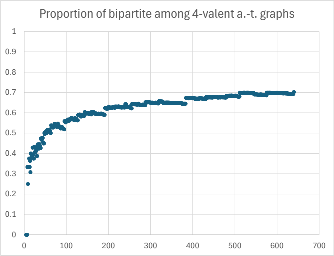


Question: Is it true that within the class of vertex-transitive (arc-transitive) cubic graphs the probability of bipartedness goes to 1 as $|V| \to \infty$?

What about for other valences?

Prevalence of bipartness





Conjecture: For each fixed $d \geq 3$, almost every connected d-valent vertex-transitive graphs is bipartite.

Note that without vertex-transitivity: For each fixed $d \geq 3$, almost every connected d-valent graphs is non-bipartite.

Main message

- Catalogues are useful for:
  - testing existing conjecture.
  - finding patterns and posing conjecture .
- They test our understanding of the theory.
- They also motivate new theoretical research.