Datasets of regular maps and their skeletons

Primož Potočnik (University of Ljubljana)

joint work with Marston Conder

partially based on earlier joint work with

Pablo Spiga and Gabriel Verret

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What is a map

There are numerous ways to define a map. We will use the following two:

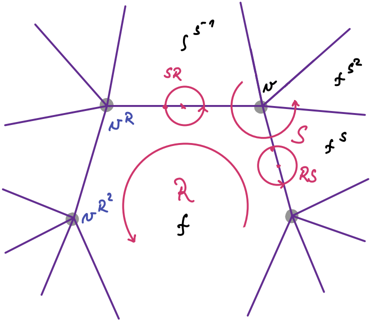
- As an embedding of a graph onto surface.
- In terms of incidence geometry, where one specifies vertices, edges, faces and incidence between them.
- Both have issues. We will sweep them under the carpet.
- See https://rotarymaps.graphsym.net for the datasets of maps from this talk.

Rotary maps

Fix a vertex v lying on a face f of a map M. If M admits symmetries:

- R, which acts as a one-step rotation of f; and
- S, which acts as a one-step rotation around v,

then M is a rotary map and $G = \langle R, S \rangle$ is the rotation group of M.



- $(RS)^2 = 1$;
- Order(R) = “face-length”;
- Order(S) = “valence”;
- G transitive on:
  - faces, vertices, arcs.
  - incident vertex-face pairs.
- has at most two orbits on incident v-e-f pairs (flags).

ROTARY MAPS ON ORIENTABLE SURFACES

Orientable maps from a group

Suppose $G = \langle R, S \mid (RS)^2, \ldots \rangle$. We say that (G, R, S) is a 2^*-triple. Define a map by:

- vertices: $G/\langle S \rangle$;
- faces: $G/\langle R \rangle$;
- edges: $G/\langle RS \rangle$;
- incidence: Non-empty intersection of cosets.

This determines a rotary map $\mathrm{OrM}(G; R, S)$ on an orientable surface.

Lemma

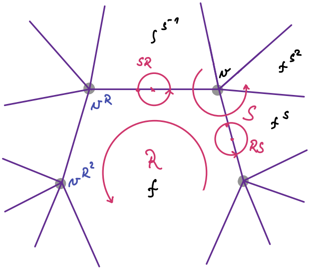
Every rotary map on an orientable surface is isomorphic to $\mathrm{OrM}(G; R, S)$ for some 2^*-triple (G, R, S).

A closer look

We have: Orientable rotary maps $\longleftrightarrow$ 2^*-triples

We want: Iso classes of ORM $\longleftrightarrow$ Iso classes of 2^*-triples

- $\mathcal{M} \cong \mathcal{M}' \Leftrightarrow$
  $\exists\, \mathcal{S} \xrightarrow{\approx} \mathcal{S}'$ preserving the graph.
- $(G, R, S) \cong (G', R', S') \Leftrightarrow$
  $\exists\, G \xrightarrow{\approx} G',\ R \mapsto R',\ S \mapsto S'$
- $\mathcal{M} \cong \mathcal{M}'$? $\Rightarrow$?
  $(G, R, S) \cong (G', R', S')$.



Therefore:
Iso classes of 2^*-triples correspond to iso classes of oriented rotary maps.

Constructing a census of rotary oriented maps

Task: Find all rotary maps on orientable surfaces with at most N edges.

- Suffices to find all 2^*-groups up to order $2N$.
- For each such group G:
  Find all generating pairs R, S, (satisfying $(RS)^2 = 1$) up to equivalence under $\mathrm{Aut}(G)$.

Classical approach:

- Use the database of small groups (not practical for $N \geq 512$).
- Compute finite quotients of the universal group $\langle R, S \mid (RS)^2 \rangle$

Alternative approach (PS, GV, PP, 2014)

Let (G, R, S) be a 2^*-triple.

Suppose $\exists N : 1 \neq N^{\mathrm{ab}} \lhd G$. WLOG: N minimal, $\cong \mathbb{Z}_p^d$.

- Then $(G/N, \bar{R}, \bar{S})$ is a 2^*-group or G/N is cyclic.

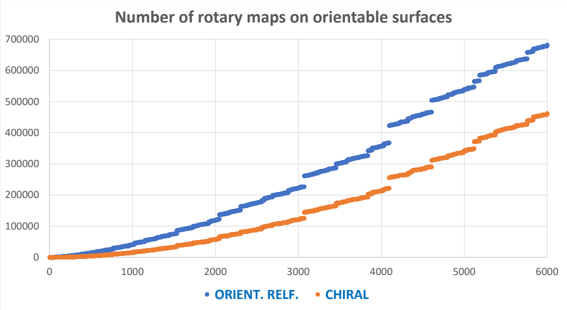
If G has no non-trivial normal abelian subgroups, then it is semi-simple.

Strategy:

- Start with cyclic and semisimple groups.
- Inductively compute minimal extensions by elementary abelian grps.
- At each step, find all generating pairs (R, S) of the group G, $(RS)^2 = 1$, up to conjugacy in $\mathrm{Aut}(G)$.

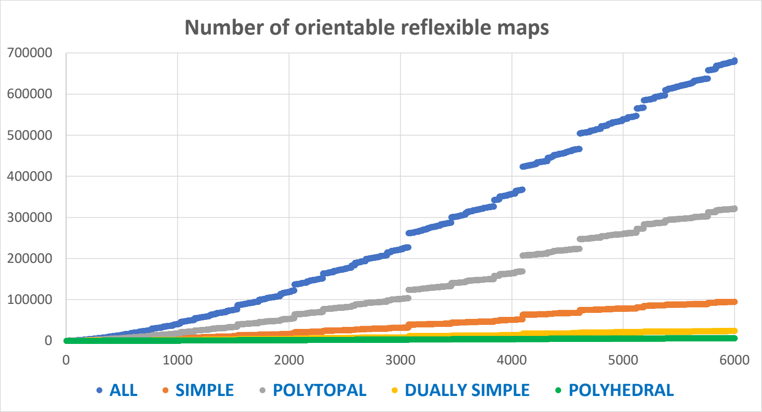
$(2,*)$-groups up to order $12{,}000$

- There are 397,385 $(2,*)$-groups of order up to $12{,}000$;
- yielding $1{,}144{,}876$ $(2,*)$-generating pairs;
- and thus $1{,}144{,}876$ orientable rotary maps with $\leq 6{,}000$ edges.
- Our of these $682{,}304$ reflexible and $462{,}572$ chiral .

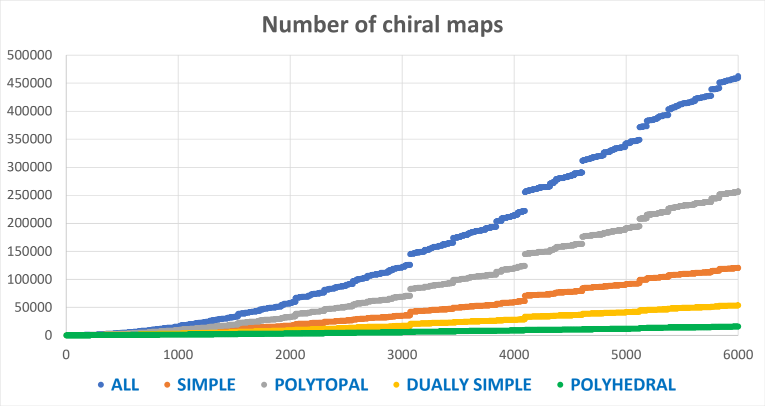


- Asymptotically: $m^{b\log(m)} \leq f(m) \leq m^{c\log(m)}$.

More charts: "non-degenerate" orientable reflexible maps



More charts: “non-degenerate” chiral maps



NON-ORIENTABLE MAPS

Reflexible oriented maps

Let (G, R, S) be a 2^*-group.

If $\mathrm{OrM}(G, R, S)$ is reflx, then $\exists\, \tau \in \mathrm{Aut}(G):\ R^\tau = R^{-1}, S^\tau = S^{-1}$.

Let $\hat{G} = G \rtimes \langle \tau \rangle$, $a := R\tau$, $b := \tau$, $c := \tau S$.

Then: $\hat{G} = \langle a, b, c \rangle$, $a^2 = b^2 = c^2 = (ac)^2 = 1$; $R = ab$, $S = bc$.

Note that: $\langle ab, bc \rangle = G \neq \hat{G}$.

Define a map $\mathrm{ReflM}(\hat{G}; a, b, c)$:

- vertices: $\hat{G}/\langle b, c \rangle$;
- faces: $\hat{G}/\langle a, b \rangle$;
- edges: $\hat{G}/\langle a, c \rangle$;
- incidence: Non-empty intersection of cosets.

$$\mathrm{ReflM}(\hat{G}, a, b, c) \cong \mathrm{OrM}(G, R, S).$$

Converse

What if we start with $\hat{G} = \langle a, b, c \mid a^2, b^2, c^2, (ac)^2, \ldots \rangle$.
Let $R = ab$, $S = bc$, $G = \langle R, S \rangle$.

- If $G \neq \hat{G}$, then $\mathrm{ReflM}(\hat{G}, a, b, c) \cong \mathrm{OrM}(G, R, S)$.
- If $G = \hat{G}$, then $\mathrm{ReflM}(\hat{G}, a, b, c)$ is still a map, but on an non-orientable surface.

Important: Every rotary map on non-orientable surface is isomorphic to some $\mathrm{ReflM}(G, a, b, c)$ s.t. $\langle ab, bc \rangle = G$.

Non-orientable rotary maps

To find all non-orientable maps on up to N edges, one needs to find all quadruples (G, a, b, c) s.t.:

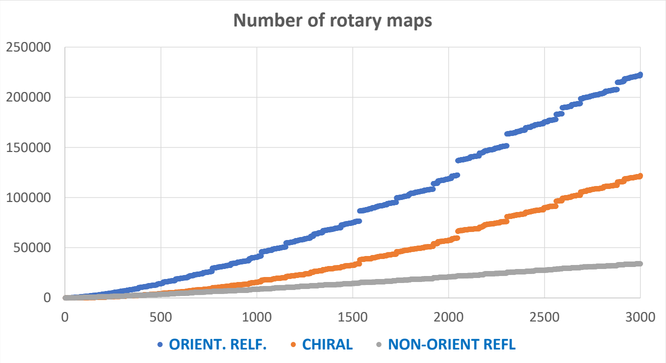
$$G = \langle a, b, c \rangle \ = \ \langle ab, bc \rangle, \ \ a^2 = b^2 = c^2 = (ac)^2 = 1, \ \ |G| \leq 4N.$$

To do that, note that (G, ab, bc) is a 2^*-group , and thus:

- Go through all 2^*-triples (G, R, S), for $|G| \leq 4N$.
- For each, find all involutions $t \in G$ such that $R^t = R^{-1}$ and $S^t = S^{-1}$. If there is one, then there are as many as there are central involutions in G.
- For each t, define $a := Rt, b := t, c := tS$.

Number of non-orientable rotary maps

As a result: All non-orientable (and thus all rotary) maps with at most $3{,}000$ edges are known.



ROTARY MAPS UP TO A GIVEN GENUS

Genus

Consider an oriented rotary map of type $\{p, q\}$. WLOG: $q \leq p$ (duality).

Bound on $|G|$ for hyperbolic surfaces:

$$\begin{aligned} |G| &= \beta(p,q)(g-1), \\ \beta(p,q) &= \frac{4pq}{pq - 2(p+q)} = \frac{4q}{q - 2 - 2\frac{q}{p}} \leq 84 \end{aligned}$$

with 84 achieved for $\{p, q\} = \{7, 3\}$.

Our goal: All oriented rotary maps of genus $\leq 1{,}501$.

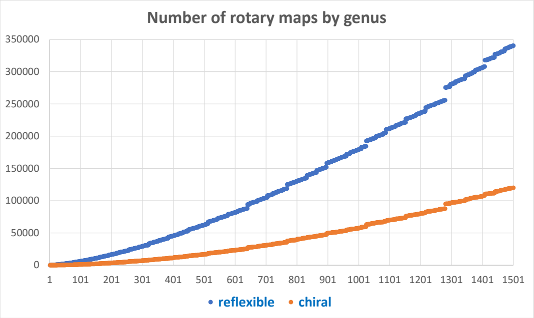
- For $q = 3$: $|G| \leq \frac{12p}{p-6}(g-1) \rightarrow 12(g-1)$ (decreasing)
  - We already had $|G| \leq 30{,}000$. This takes care of $p \geq 15$.
  - We used LINS to compute quotients of $\langle R, S \mid R^p = S^3 = (RS)^2 = 1 \rangle$ of order $\leq 1{,}500\frac{12p}{p-6}$ for $7 \leq p \leq 14$.

Oriented rotary maps of genus $\leq 1{,}501$

- For $q = 4:$ $\quad |G| \leq \frac{16p}{p-8}(g-1)$.
  - We used the extension method to compute all $(2,4)$-generated groups of order at most $15{,}000$, thus dealing with $p \geq 20$.
  - We used LINS to deal with the cases $5 \leq p \leq 19$.
- We used similar combined approach for $q = 5, 6, 7$.
- For $q \geq 8:$ $\quad |G| \leq \frac{4q}{q-4}(g-1) \leq 8 \cdot 1{,}500 = 12{,}000$

Oriented rotary maps of genus $\leq 1{,}501$

All oriented rotary maps of genus g, $2 \leq g \leq 1{,}501$, are now known. There are $580{,}540$ of them, out of which $340{,}432$ are reflexible and $240{,}108$ are chiral.

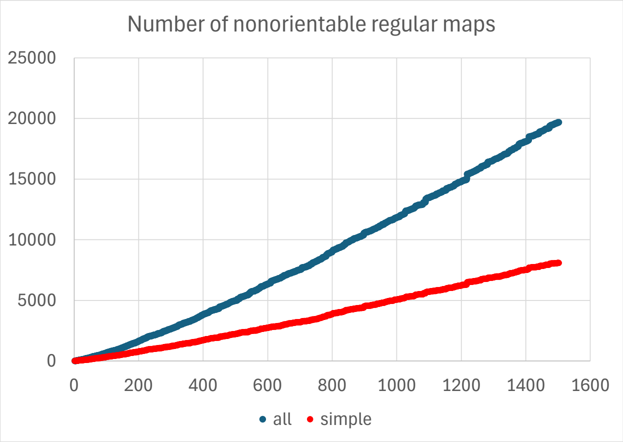


Jumps at:

$1281 = 1 + 2^8 \cdot 5,\ 1409 = 1 + 2^7 \cdot 11,\ 1025 = 1 + 2^{10}$,
$769 = 1 + 2^8 \cdot 3,\ 897 = 1 + 2^7 \cdot 7.$

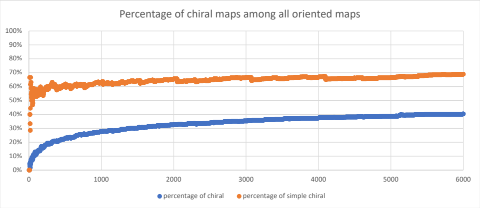
Non-orientable rotary maps of given genus

Also: All rotary maps on non-orientable surfaces of genus $1{,}502$ are known. ($19{,}696$)

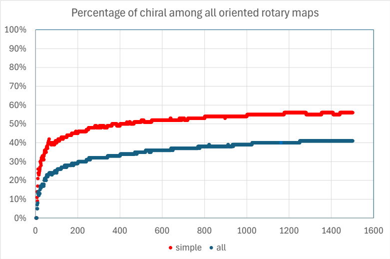


WHAT TO DO WITH ALL THIS?

Percentage of chiral maps (do they prevail asymptotically)?

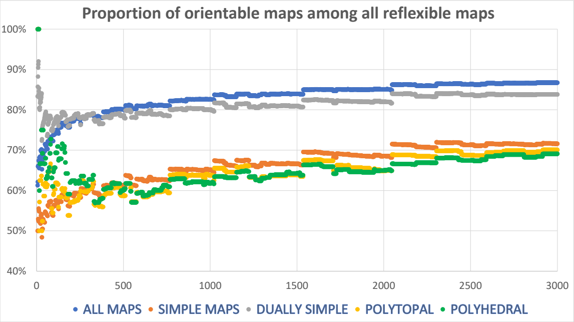


Percentage of chiral maps within all rotary maps, up to given NUMBER OF EDGES



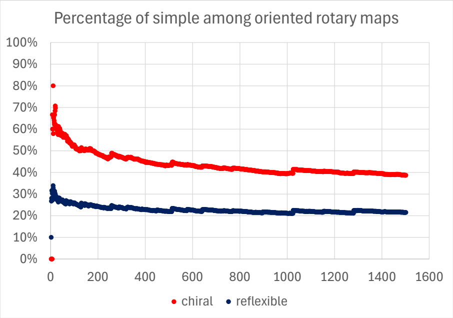
Percentage of chiral maps within all rotary maps, up to given GENUS

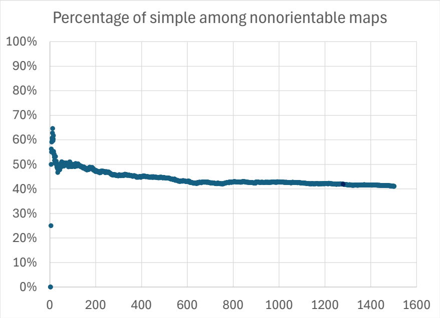
"Orientable reflexible" vs. "all non-orientable"



Question: Does the proportion of orientable maps among all reflexible maps up to given number of edges tend to 1?

Percentage of simple maps





Question: Does the proportion of simple maps among all rotary maps up to given genus tend to 0?

Finding example, testing and posing conjectures

- We found three orientable surfaces (of genera 392, 866, 1004) that do not support any simple rotary map. Previously believed not to exist.
- Steve Wilson asked the following question: Does there exist a graph that embeds as a chiral map, as well as a reflexible map both on non-orientable as well as orientable surface. With Isabel Hubard and Primož Šparl, we found an infinite family of such examples.
- Mark Ellingham asked: Can all 15 types of combinations of orientability-biparteness conditions of Wilson mates be realised in highly symmetrical maps? YES , even among reflexible (fully regular) rotary maps.

How to present and maintain datasets of objects?

- Enough data should be made public, so that anybody can reconstruct the objects.
- If possible, the format should be compatible with several platforms (magma, gap, sage, etc)
- A spreadsheet with the summary of some basic properties of the objects is desirable.
- The dataset needs to be simple to maintain, or a long-term maintenance support should be ensured.

How to credit the work

- Compiling a census takes a lot of time (human and computer).
- They are often difficult to publish in good journals (and what would one write there).
- Once the raw data is computed, there is still a lot of work to present it nicely. How to credit that?