

Cycle regularity of cubic vertex-transitive graphs

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JOINT WORK WITH

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Combinatorial homogeneity vs. symmetry

Graphs may possess different types of **combinatorial homogeneity**.

- **regularity**: all vertices have the same number of neighbours;
- **strong regularity**: every edge lies on the same number of triangles and **the same for the complement**.
- **walk regularity**: the number of closed walks (of every fixed length) starting at a vertex does not depend on the vertex.

Combinatorial homogeneity follows from **transitivity**.

- vertex-transitivity \Rightarrow **regularity** and **walk regularity**
- edge-transitivity of the graph and its complement \Rightarrow **strong-regularity**.

HOWEVER:

Transitivity (almost) never follows from **homogeneity**.

... almost never ...

- A **connected 2-regular graph** is **vertex-transitive**;
(because it is a cycle).
- **Platonic solids**: If X is a *regular convex polyhedron* (constant valence, all faces congruent to a fixed regular polygon), then $\text{Aut}(X)$ acts transitively on **vertices**, **edges**, **faces**, **flags**.
- **Maps of type $\{6, 3\}$** on orientable surface are **vertex-transitive**.

Challenge: Find other instances of **combinatorial homogeneity** that implies **algebraic symmetry**.

Cycle regularity

- For $k \in \mathbb{N}$, $e \in E(\Gamma)$:

$c_k(e)$ = number of k -cycles that pass through e .

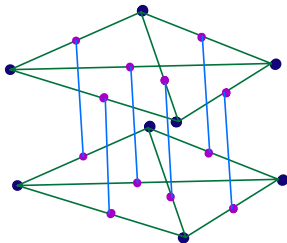
- Γ is k -cycle regular if $c_k(e)$ is constant for all $e \in E(\Gamma)$.
- If Γ is k -cycle regular for all k , then it is cycle-regular.

Clearly: Edge-transitivity \Rightarrow cycle-regularity.

QUESTION: Does cycle-regularity imply edge-transitivity?

Of course not

A small counterexample (provided to us by Royle and McKay):



20 vertices, 2 orbits on edges, 2 orbits on vertices,
3-regular (cubic), girth = 6.

A question of Fouquet and Hahn

Question (Fouquet and Hahn, 2001):

Is every cycle-regular and **vertex-transitive** also **edge-transitive**?

NO! Answered by Marston Conder and Jin-Xin Zhou (JCTB'23):

Theorem. The **line graph** of a **cubic locally-2-arc-transitive not vertex-transitive** graph is **cycle-regular, vertex-transitive** but **not edge-transitive**.

Corollary. There are infinitely many **tetravalent** cycle-regular, vertex-transitive but not edge-transitive graphs of girth 3.

The peculiar case of cubic vertex-transitive graphs

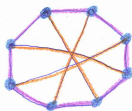
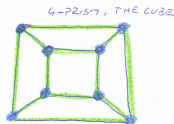
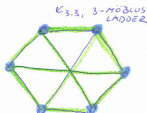
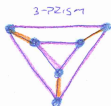
So, we have tetravalent examples. Can one find examples of valence 3?

Question (Conder, Jin-Xin): In the realm of **connected cubic vertex-transitive graphs**:

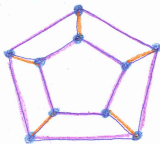
Cycle-regularity $\stackrel{?}{\Rightarrow}$ **edge-transitivity**.

First reaction: SURELY **NOT**!

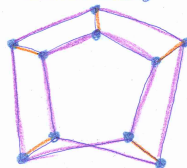
Intermezzo: Census of cubic vertex-transitive graphs



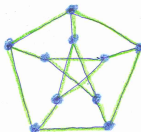
4-MÖBIUS LADDER,
 $K_{4,4} - 4K_2$,
WAGNER GRAPH



5-PRISM



5-MÖBIUS LADDER

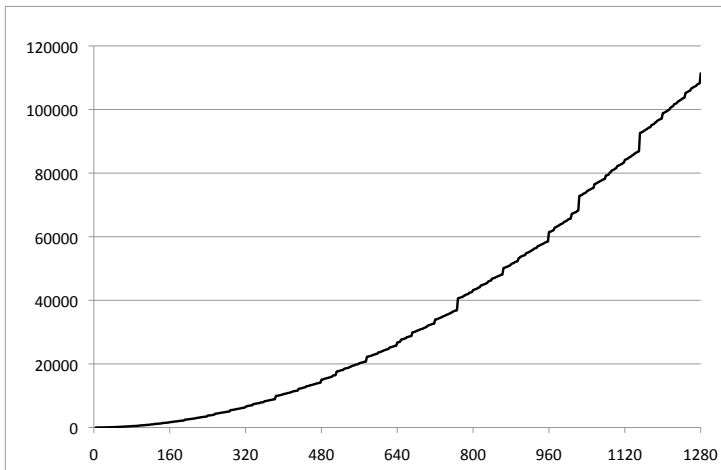


THE PETERSEN GRAPH

Intermezzo: Census of cubic vertex-transitive graphs

- All cubic vertex-transitive graphs up to order 1280 are known. (Spiga, Verret, PP)
- There are 111 360 of them.
- Only 482 of them are edge-transitive.
- **Surprise:** None of the 110 878 non-edge-transitive ones is cycle-regular!

The number of cubic vertex-transitive graphs



Current status

Theorem 1. Every cubic vertex-transitive cycle-regular graph on at most 1280 vertices is edge-transitive.

Theorem 2. Every cubic (vertex-transitive) cycle-regular graph of girth at most 5 is edge-transitive.

Theorem 3. (Verret, PP, 2023) Every cubic vertex-transitive cycle-regular graph of girth 6 is edge-transitive.

Theorem 2: Girth at most 5

As observed independently by many authors:

Theorem. Let Γ be a cubic graph of girth $g \leq 5$. If Γ is g -cycle-regular, then:

- $g = 3$ and $\Gamma \cong K_4$;
- $g = 4$ and $\Gamma \cong K_{3,3}$ or the cube Q_3 ;
- $g = 5$ and Γ is either the Petersen graph or the Dodecahedron.

In particular, Γ is **vertex- and edge-transitive**.

This takes care of girth at most 5.

Theorem 3: Girth 6

The starting point is the following classification:

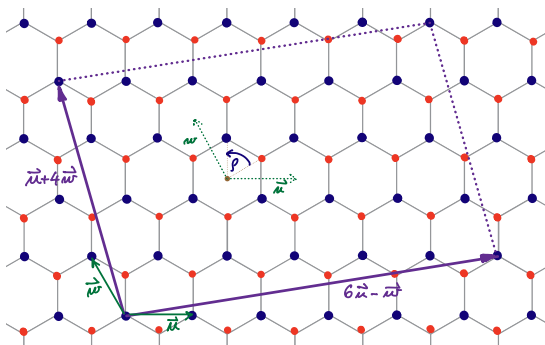
Theorem. (P., Vidali, 2022) Let Γ be a cubic vertex-transitive graph of girth 6. If Γ is 6-cycle regular, then:

- $c_6(e) = 8$ and Γ is the Heawood graph (on 14 vertices),
- $c_6(e) = 6$ and Γ is the Möbius-Kantor graph (on 16 vertices),
- $c_6(e) = 4$ and Γ is
 - the Pappus graph (on 18 vertices); or
 - the Desargues (on 20 vertices),
- $c_6(6) = 2$ and Γ underlies a map of type $\{6, 3\}$ on the torus.

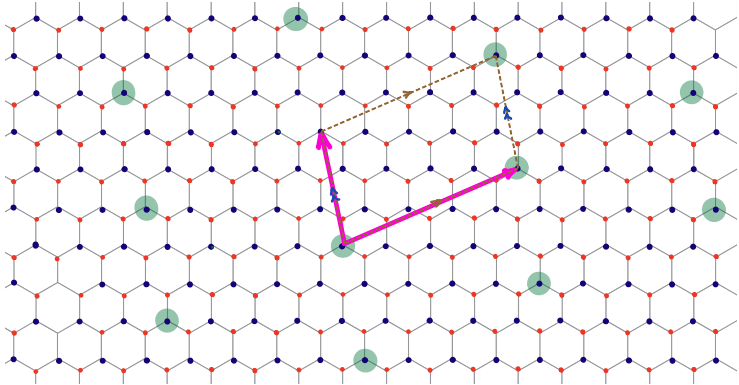
Hence, it suffices to consider maps of type $\{6, 3\}$ on the torus.

Lifting to the Euclidean plane

- Γ ... cubic graph, embedded onto torus T as a $\{6, 3\}$ -map.
- Consider the universal covering projection $\wp: \mathbb{R}^2 \rightarrow T$.
- Γ lifts to a hexagonal tessellation of \mathbb{R}^2 .
- **Fibres** are orbits of some group of translations H .



Trivial vs. non-trivial cycles



Cycles in Γ are of two types:

- **Trivial:** Lift to cycles in \mathcal{H} .
- **Non-trivial:** Lift to paths between vertices in the same fibre.

Short cycles

Important parameter:

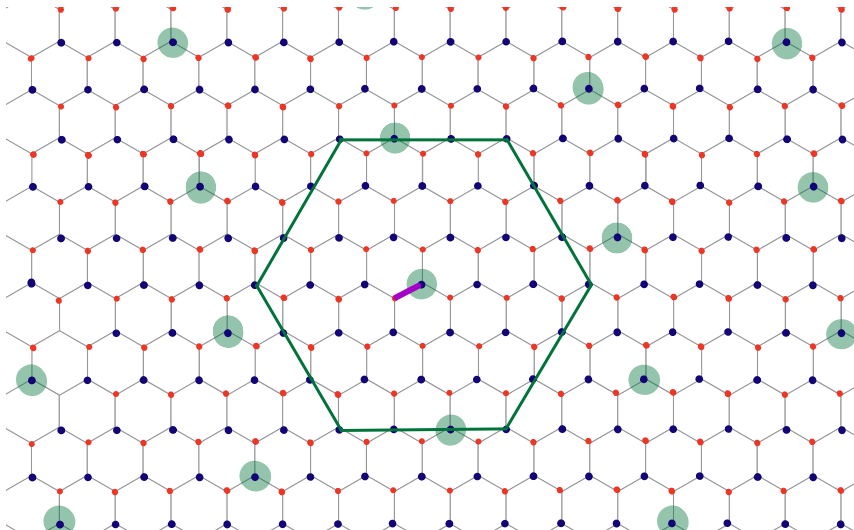
$$\begin{aligned}d_{\min} &= \text{shortest distance between two vertices in } L \\ &= \text{length of shortest non-contractible cycle in } \Gamma.\end{aligned}$$

Observations:

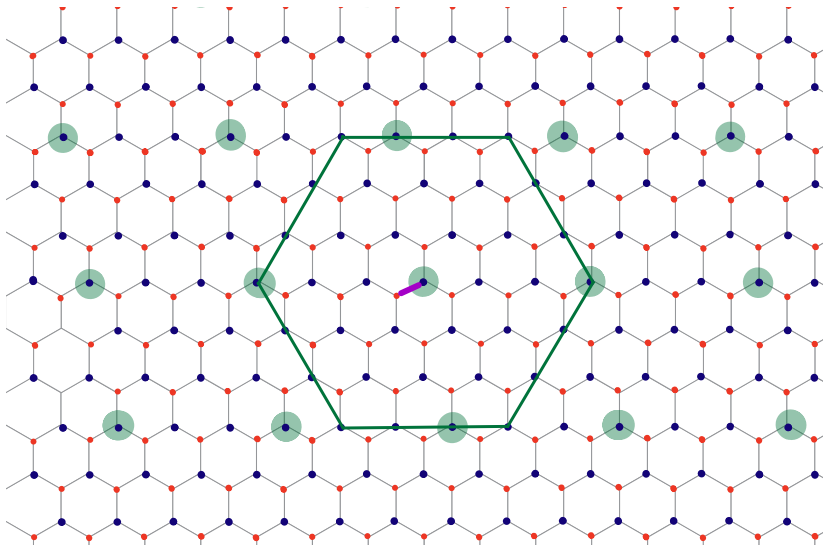
- Every cycle of length $< d_{\min}$ is trivial.
- Γ is ℓ -cycle-regular for all $\ell < d_{\min}$.
- Critical cycle lengths: $\ell = d_{\min}$
- In particular, we need to consider the number of ℓ -paths, $\ell = d_{\min}$ and $d_{\min} + 2$ between two vertices in a fibre.
- First is easy. The second involves solving:

$$L(n, k) = L(n-1, k-1) + L(n-1, k) + \binom{n-1}{k-2} + \binom{n-1}{k+1}.$$

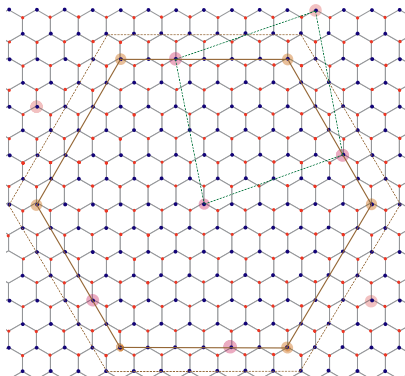
$$|D| = 2$$



$$|D| = 4$$

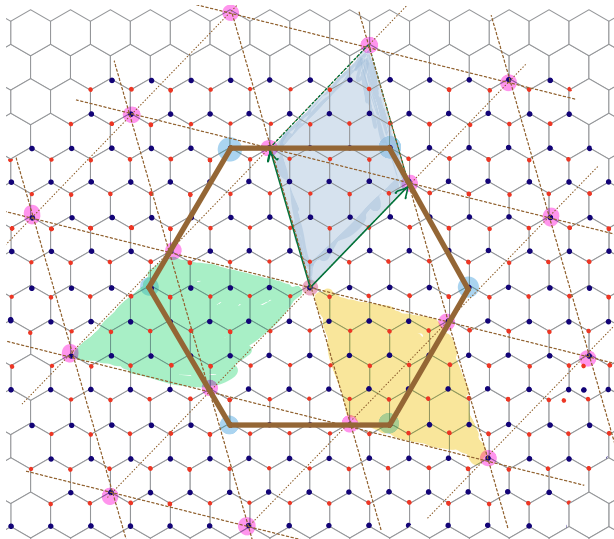


d_{\min} -cycle regular examples



- Infinite family of graphs $\Gamma(s)$, $s \geq 1$, that are k -cycle regular for all $k \leq \frac{3}{2}\sqrt{|V|}$.
- Smallest example: $\Gamma(s) = \text{Möbius-Kantor graph}$ (which is edge-transitive).

Symmetric case. $|D| = 6$



Conclusion

Within the realm of cubic vertex-transitive graphs:

- We proved: If $\text{girth} \leq 5$, then:
cycle-regular \Rightarrow edge-transitive.
 - Up to girth 5, there are only finitely many cubic VT graphs with every edge on the same number of girth cycles — they are all edge-transitive.
- We also proved: If $\text{girth} = 6$, then: cycle-regular \Rightarrow edge-transitive.
 - Reduction to maps with faces of length 6;
 - Analysis of the maps.

Higher girth

Girth 7 (still cubic VT):

- It seems that reduction to maps is possible:

Computational data: if every edge on the same number of 7 cycles, then either Coxeter graph or a $\{7, 3\}$ -map.

- An easy group theory argument: Every vertex-transitive $\{7, 3\}$ -map is edge-transitive.

Girth = 8:

- Reduction to maps might be possible
- No idea how to deal with maps.